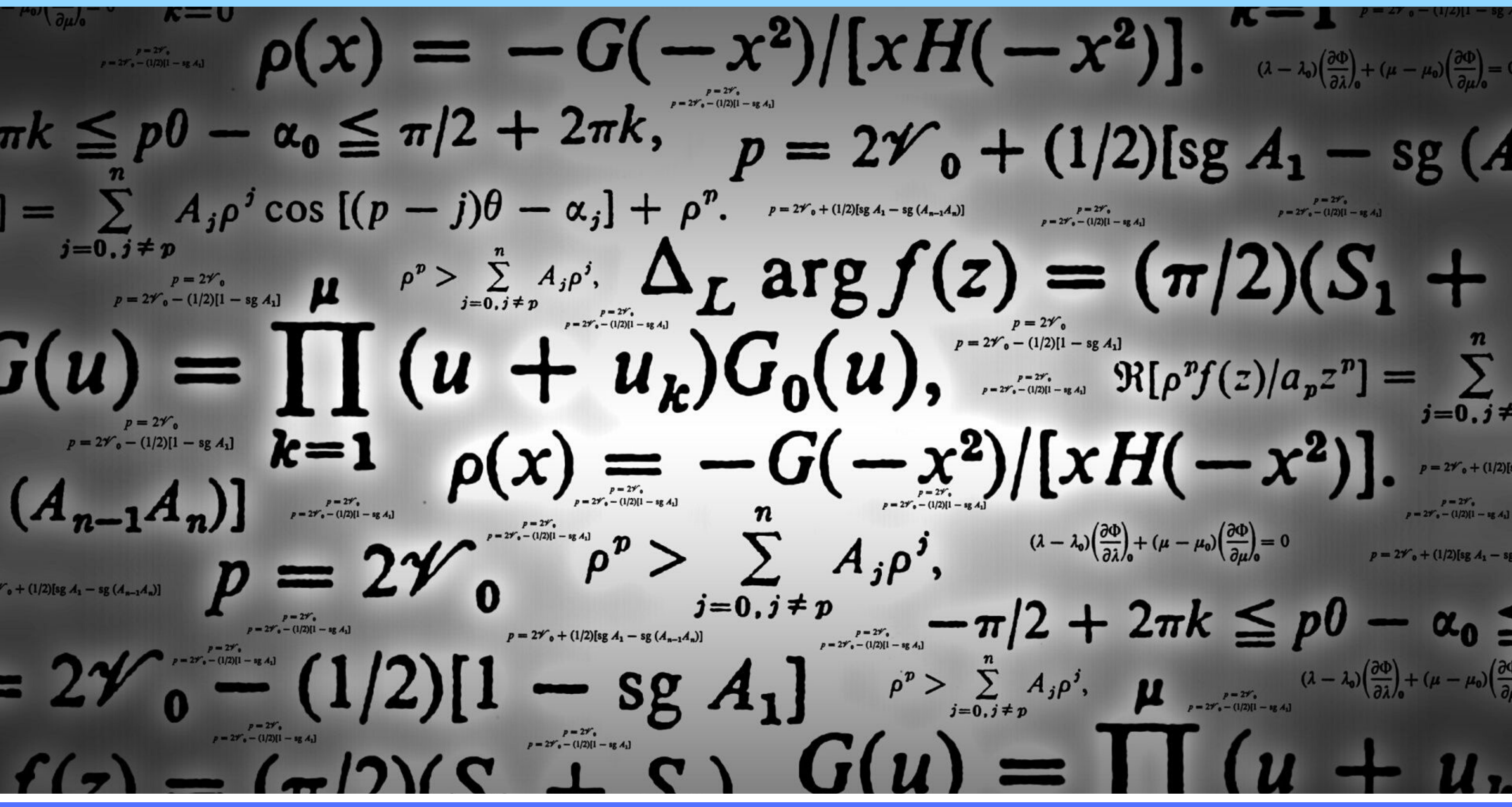
FUNCTIONS AND GRAPHS

REVISION BOOKLET

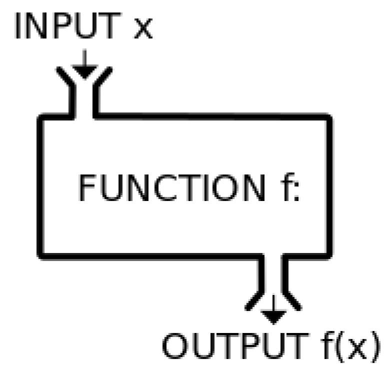


YEAR 11 MATHEMATICS

1. Functions

Functions are the special class of relation or we can say that special types of relations are called as functions. It is the important concept used frequently in mathematics.

A function is like a machine which gives unique output for each input that is fed into it. But every machine is designed for certain defined inputs for e.g. a washing machine is designed for washing cloths and not the wood. Similarly, the functions are also defined for certain inputs which are called as its **Domain** and corresponding outputs are called **Range**.



Let A and B be two sets and let there exist a manner or rule or correspondence 'f' which associates every element of A to a special or unique element in B, then f is called a **Function** or **Mapping** from A to B. It is denoted by symbol.

$$f: (A, B) \text{ or } f: A \to B \text{ or } A \xrightarrow{f} B$$

Which reads 'f is a function from A to B' or' f maps A to B.

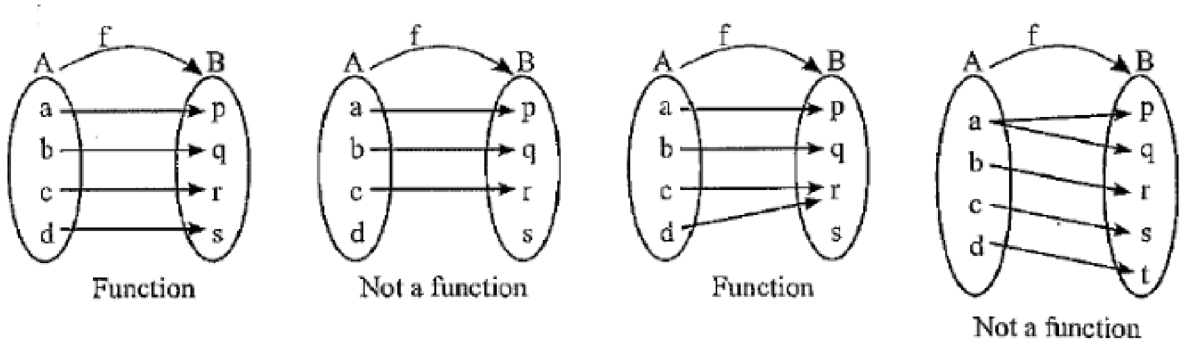
- If an element **a** ∈ **A** is associated with an element **b** ∈ **B** then b is called '**Image** of **a** under **f'** or 'the ***f image*** of ***a***' or 'the value of the function at ***a'***. Also a is called the ***pre – image*** of ***b*** or argument of ***b*** under the function we write it as

f : (a, b) or f : a → b or b = f(a)

- A relation from a set A to a set B is called as the **Function** if it satisfies the below conditions:

- All the elements of A should be mapped with the elements of B.
- Every element in A has to correspond to a unique element in B.

Thus, the ordered pairs of *f* must satisfy the property that each element of A appears in some ordered pair and no two ordered pairs have same first elements. See the below figures to understand the above points.



- **Note**: Every function is a relation but every relation is not necessarily a function.

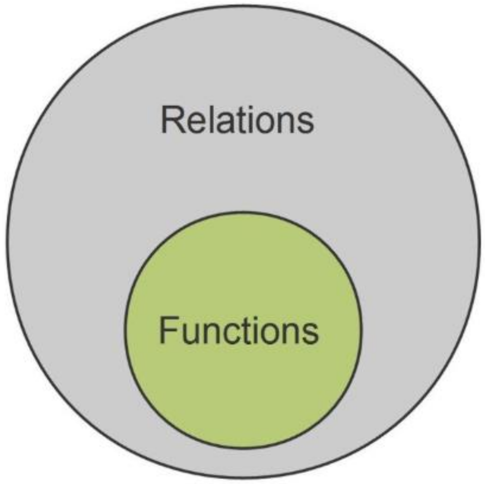


Figure 1 Every function is a relation but every relation is not necessarily a function.

- **Vertical Line Test**

- The **vertical line test** is a method that is used to determine whether a given relation is a function or not. The approach is rather simple. Draw a vertical line cutting through the graph of the relation, and then observe the points of intersection.

- The vertical line test supports the definition of a function. That is, every *x*-**value** of a function must be paired to a single *y*-**value**. If we think of a vertical line as an infinite set of *x*-**values**, then intersecting the graph of a relation at exactly one point by a vertical line implies that a single *x*-**value** is only paired to a unique value of *y*.

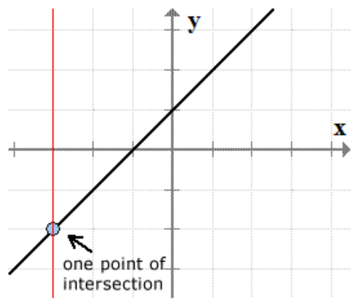
- In contrary, if the vertical line intersects the graph more than once this suggests that a single *x*-**value** is being associated with more than one value of *y*. This condition causes the relation to be "disqualified" or not considered as a function.

> If a vertical line intersects the graph in all places **at exactly one point**, then the relation is a **function**.

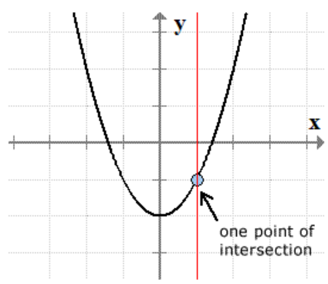
- Here are some examples of relations that are also functions because they pass the vertical line test.

Cutting or Hitting the Graph at <u>Exactly One Point</u>

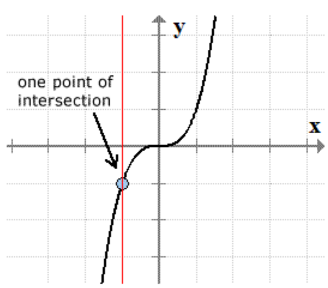
Graph of the line $f(x) = x + 1$



Graph of the quadratic function (parabola) $f(x) = x^2 - 2$



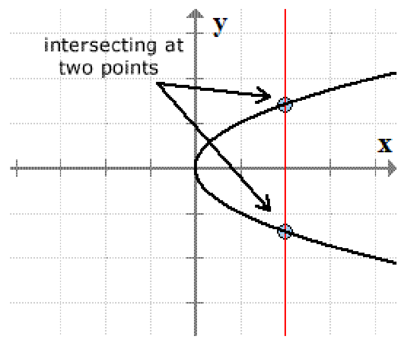
Graph of the cubic function $f(x) = x^3$



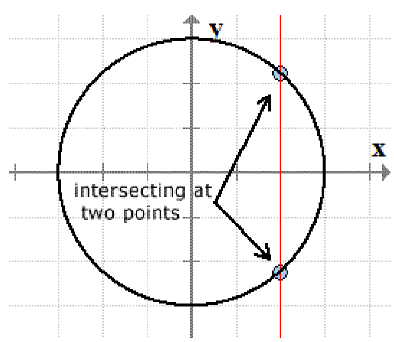
- If a vertical line intersects the graph in some places at **more than one point**, then the relation is **NOT** a function.
  Here are some examples of relations that are NOT functions because they fail the vertical line test.

Cutting or Hitting the Graph in More Than One Point

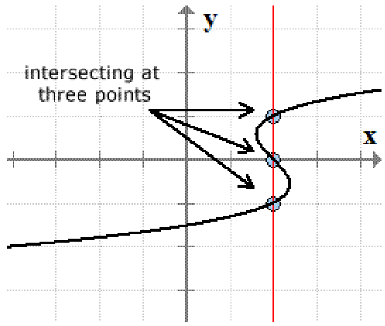
Graph of the "sideway" parabola $x = y^2$



Graph of the circle $x^2 + y^2 = 9$

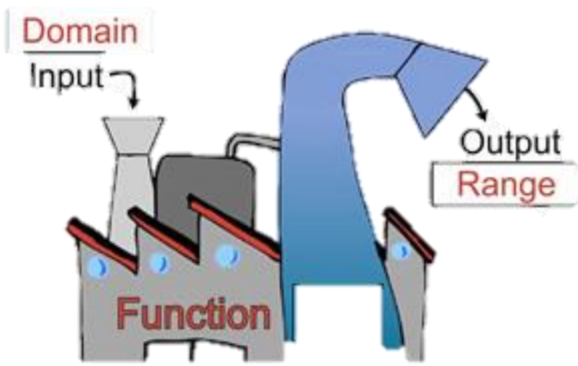


Graph of the relation $x = y^3 - y + 2$



2. Domain and Range

Under Function Basics, we saw the difference between a relation and a function.
In either case, we are dealing with relationships expressed as ordered pairs.



- All of the values that can go into a relation or function (*input*) are called the domain.
- All of the values that come out of a relation or function (*output*) are called the range. Range may also be referred to as "image".

- **Note:** that both relations and functions have domains and ranges.

- The domain is the set of all first elements of ordered pairs (*x*-coordinates).

- The range is the set of all second elements of ordered pairs (*y*-coordinates). Only the elements "used" by the relation or function constitute the range.

Example 1:

State the domain and range of the following relation:
(eye color, student's name).

A = {(blue,Steve), (green,Elaine), (brown,Kyle), (blue,Marsha), (brown,Miranda), (green, Dylan)}
State whether the relation is a function.

Solution: Domain: {blue, green, brown}. Range: {Steve, Elaine, Marsha, Miranda, Dylan}.
No, this relation is not a function. The eye colors are repeated.

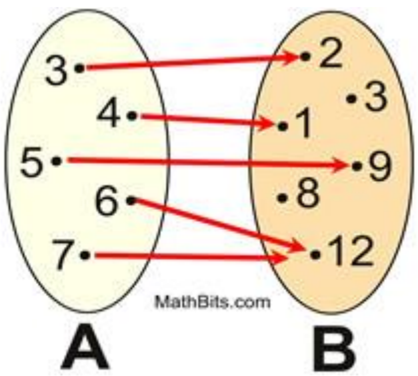
Example 2

State the domain and range of the following relation: {(1,3), (-2,7), (3,-3), (4,5), (1,-3)}.
State whether the relation is a function.

Solution: Domain: {-2, 1, 3, 4}. Range: {-3, 3, 5, 7}.
While these listings appear in ascending order, ordering is not required. Do not, however, duplicate an element.
No, this relation is not a function. The *x*-value of "1" had two corresponding *y*-values (3 and -3).

Example 3

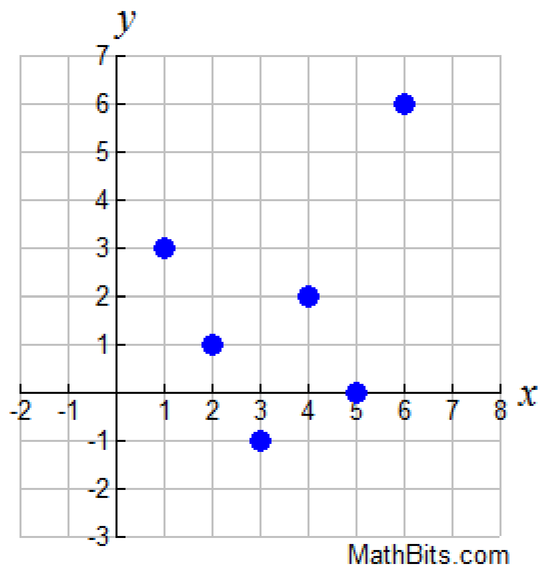
State the domain and range for the elements matched in the diagram below.
State whether the matches form a function.



Solution: Domain: {3, 4, 5, 6, 7}. Range: {1, 2, 9, 12}.
Note that the range is only the elements that were used.

Yes, the relation {(3,2), (4,1), (5,9), (6,12), (7,12)} is a function.
No *x*-value repeats.

Example 4
State the domain and range associated with the scatter plot shown below.
State whether the scatter plot is a function.



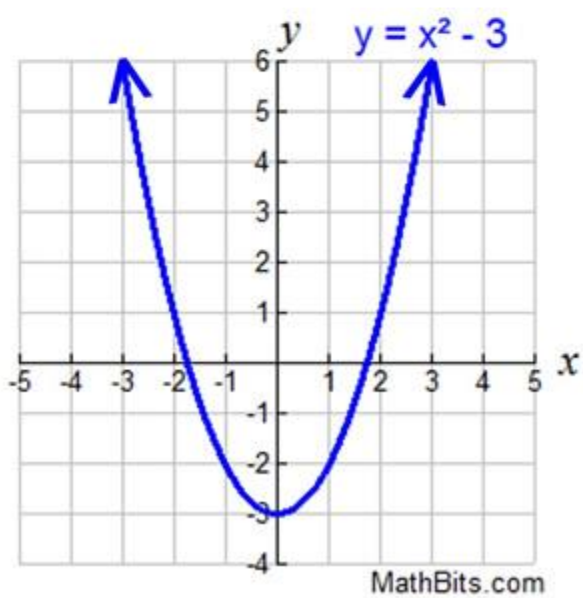
Solution: Domain: {1, 2, 3, 4, 5, 6}.
(Be careful not to simply list the domain as $1 < x < 6$, which would imply ALL values between 1 and 6 inclusive, unless you specify "*x* is an integer".)

Range: { 0, -1, 1, 2, 3, 6}

Yes, this is a function. No *x*-values repeat, and it passes the Vertical Line Test for functions.

Note: Graphs that are composed of a series of dots, instead of a connected curve, are referred to as discrete graphs. A discrete domain is a set of input values that consist of only certain numbers in an interval.

Example 5
State the domain and range associated with the graph below.
State whether this relation is a function.



Solution: Domain: $\mathbb{R}$ (all real numbers).
The arrows indicate that the graph continues off the visible grid, so assume that all real numbers are involved.

Range: $y \geq -3$ (may also be written as $\{y \in \mathbb{R} \mid y \geq -3\}$)

Yes, this relation is a function, since it passes the Vertical Line Test for functions.

Note: Graphs that are composed of a connected curve are referred to as continuous graphs.
A continuous domain is a set of input values that consists of all numbers in an interval.

Function	$y = x$	$y = \frac{1}{x}$	$y = \sqrt{x}$	$y = \frac{1}{\sqrt{x}}$
Domain	$x \in R$ *or* $(-\infty, \infty)$	$x \neq 0$ *or* $(-\infty, 0) \cup (0, \infty)$	$x \geq 0$ *or* $[0, \infty)$	$x > 0$ *or* $(0, \infty)$

Exercise

1. Find the natural domain of each function.

a $f(x) = 4x$ **b** $f(x) = 7 - 3x$ **c** $f(x) = \frac{1}{4 - x}$ **d** $f(x) = \frac{3}{2x - 1}$

e $f(x) = \sqrt{x + 4}$ **f** $f(x) = \sqrt{2x + 1}$ **g** $f(x) = \sqrt{5 - x}$ **h** $f(x) = \sqrt{4 - 2x}$

i $f(x) = \frac{1}{\sqrt{x}}$ **j** $f(x) = \frac{1}{\sqrt{x + 1}}$ **k** $f(x) = \frac{2}{\sqrt{1 - x}}$ **l** $f(x) = \frac{1}{\sqrt{2x - 3}}$

2. State the natural domain of each function.

a $f(x) = \frac{x}{\sqrt{x + 2}}$ **b** $f(x) = \frac{2}{x^2 - 4}$ **c** $f(x) = \frac{1}{x^2 + x}$

d $f(x) = \frac{1}{x^2 - 5x + 6}$ **e** $f(x) = \sqrt{x^2 - 4}$ **f** $f(x) = \frac{1}{\sqrt{1 - x^2}}$

3. Use the quadratic formula to find the roots α and β of each quadratic equation. Hence show in each case that

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

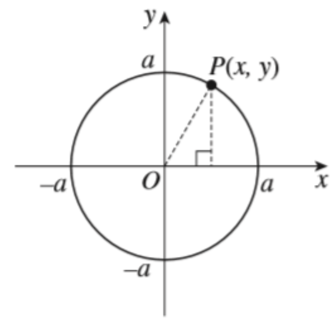
a $x^2 - 6x + 1 = 0$ **b** $x^2 - 2x - 4 = 0$ **c** $-3x^2 + 10x - 5 = 0$

4. Find the discriminant Δ = b2 − 4ac of each quadratic. Use this and the concavity to state how many zeroes the function has, without drawing its graph.

a $f(x) = x^2 + 3x - 2$ **b** $f(x) = 9x^2 - 6x + 1$ **c** $f(x) = -2x^2 + 5x - 4$

3. Circles

The equation of the circle with centre the origin and radius a can be found using Pythagoras' theorem, in the form of the distance formula.



A point $P(x, y)$ in the plane will lie on the circle if its distance from the centre is the radius a.

That is, if

$$\begin{aligned} OP &= a \\ OP^2 &= a^2 \\ (x-0)^2 + (y-0)^2 &= a^2 \\ x^2 + y^2 &= a^2. \end{aligned}$$

To put it very briefly, the equation of a circle is Pythagoras' theorem.

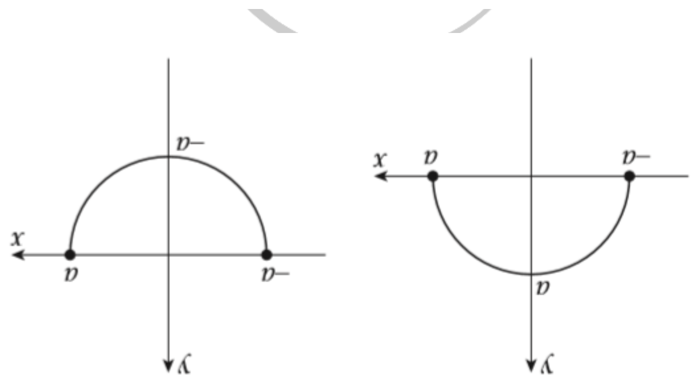
This graph fails the vertical line test, so is not a function. This can also be seen algebraically — solving the equation for y yields

$$y = \sqrt{a^2 - x^2} \qquad \text{or} \qquad y = -\sqrt{a^2 - x^2}$$

giving two values of y for some values of x.

The *positive square root* $y = \sqrt{a^2 - x^2}$, however, is a function, whose graph is the *upper semicircle* on the left below.

Similarly, the *negative square root* $y = -\sqrt{a^2 - x^2}$ is also a function, whose graph is the *lower semicircle* on the right below.



4. Asymptotic Functions (Exponential, logarithmic and Hyperbolic)

- **Direct Variation**
  - $y = kx$
  - K is a constant

- **Inverse Variation**

- $y = \frac{k}{x}$
- K is a constant

Examples:

- Direct Variation

a Garden mulch is sold in bulk, with the cost C proportional to the volume V in cubic metres. Write this algebraically.

b The shop quotes \$270 for 7.5 m^3. Find the constant of proportionality, and graph the function.

c How much does 12 m^3 cost?

d How much can I buy for \$600?

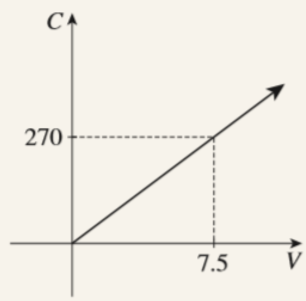
SOLUTION

a $C = kV$, for some constant k.

b Substituting the known values, $270 = k \times 7.5$

$k = 36.$

(More precisely, $k = \$36/m^3$.)



c $C = 36 \times 12$

$= \$432$

d $600 = 36V$

$V = 16\frac{2}{3}\ m^3$

- Inverse Variation

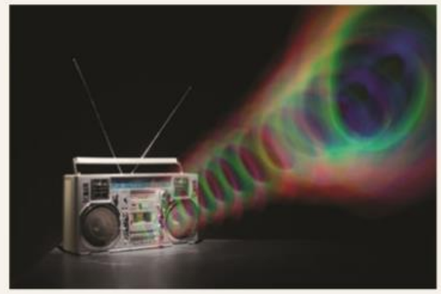
a The wavelength λ in metres of a musical tone is inversely proportional to its frequency f in vibrations per second. Write this algebraically.

b The frequency of middle C is about 260 s^{-1} ('260 vibrations per second'), and its wavelength is about 1.319 m. Find the constant of proportionality.

c Find the wavelength of a sound wave with frequency 440 s^{-1}.

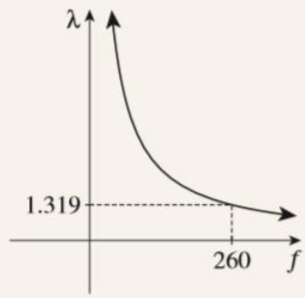
d Find the frequency of a sound wave with wave length 1 m.

e What is the approximate speed of sound in air, and why?



SOLUTION

a $\lambda = \frac{k}{f}$, for some constant k.



b Substituting the known values, $1.319 = \frac{k}{260}$

$$k \doteqdot 343.$$

(More precisely, $k = 343\,\text{m s}^{-1}$.)

c $\lambda = \frac{k}{440}$

$\doteqdot 0.779\,\text{m}$

d $\lambda = \frac{k}{f}$

$f \doteqdot 343\,\text{s}^{-1}$

e About $343\,\text{m s}^{-1}$, because 343 waves, each 1 metre long, go past in 1 second.

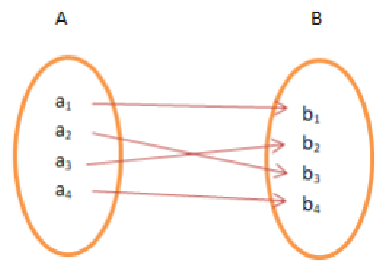
5. Horizontal line test

The graph above is called a many-to-one function, because many x-values all map to the one y-value. To formalize this, we introduce the horizontal line test. This test is the companion of the vertical line test — the two definitions simply exchange the words 'vertical' and 'horizontal'.

- **Vertical line test:**
  No vertical line meets the graph more than once.
- **Horizontal line test:**
  No horizontal line meets the graph more than once.

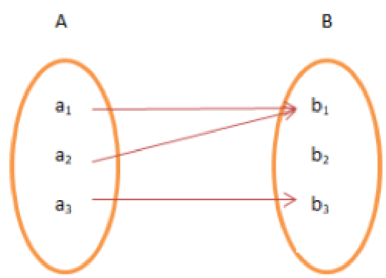
One to One Function

A function f: A → B is One to One if for each element of A there is a distinct element of B. It is also known as Injective. Consider if $a_1 \in A$ and $a_2 \in B$, f is defined as f: A → B such that f (a_1) = f (a_2)



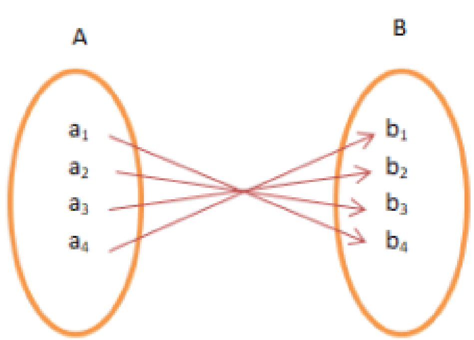
Many to One Function

It is a function which maps two or more elements of A to the same element of set B. Two or more elements of A have the same image in B.



One – One and Onto Function

A function, f is One – One and Onto or Bijective if the function f is both One to One and Onto function. In other words, the function f associates each element of A with a distinct element of B and every element of B has a pre-image in A.



VERY IMP

Type	Vertical line test	Horizontal line test
One-to-one	Passes (so a function)	Passes
Many-to-one	Passes (so a function)	Fails
One-to-many	Fails (so not a function)	Passes
Many-to-many	Fails (so not a function)	Fails

Exercise II

1. Explain, with an example using a y-value, why each function is many-to-one.

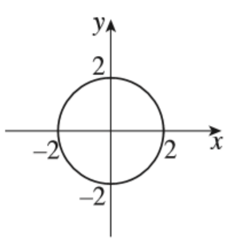
i $y = x^2 - 4$ ii $y = |x - 3|$ iii $y = (x - 1)x(x + 1)$ iv $y = x^4 + 1$

2. Classify each relation as one-to-one, many-to-one, one-to-many or many-to-many

a $y = 4$ b $x = -3$ c $x + y = 0$ d $x^2 + y^2 = 0$

3. Write down the zeroes of each cubic, use a table of values to test its sign, then sketch it, showing the y-intercept. a y = (x − 1)(x − 3)(x − 5) b y = −3(x + 4)x(x − 2) c y = 2x2(3 − x)

4. Write down the equation of each circle

a


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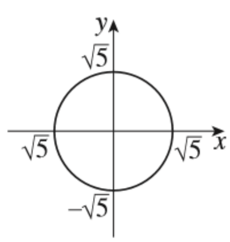
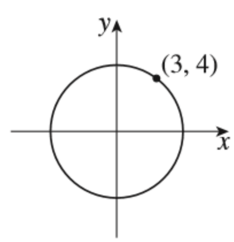
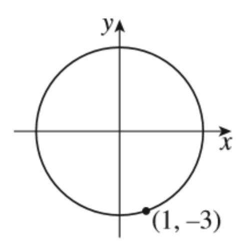


c


d


5. Sketch each semicircle, and state the domain and range.

a $y = \sqrt{4 - x^2}$ b $y = -\sqrt{4 - x^2}$ c $y = -\sqrt{1 - x^2}$

d $y = \sqrt{\frac{25}{4} - x^2}$ e $y = -\sqrt{\frac{9}{4} - x^2}$ f $y = \sqrt{0.64 - x^2}$

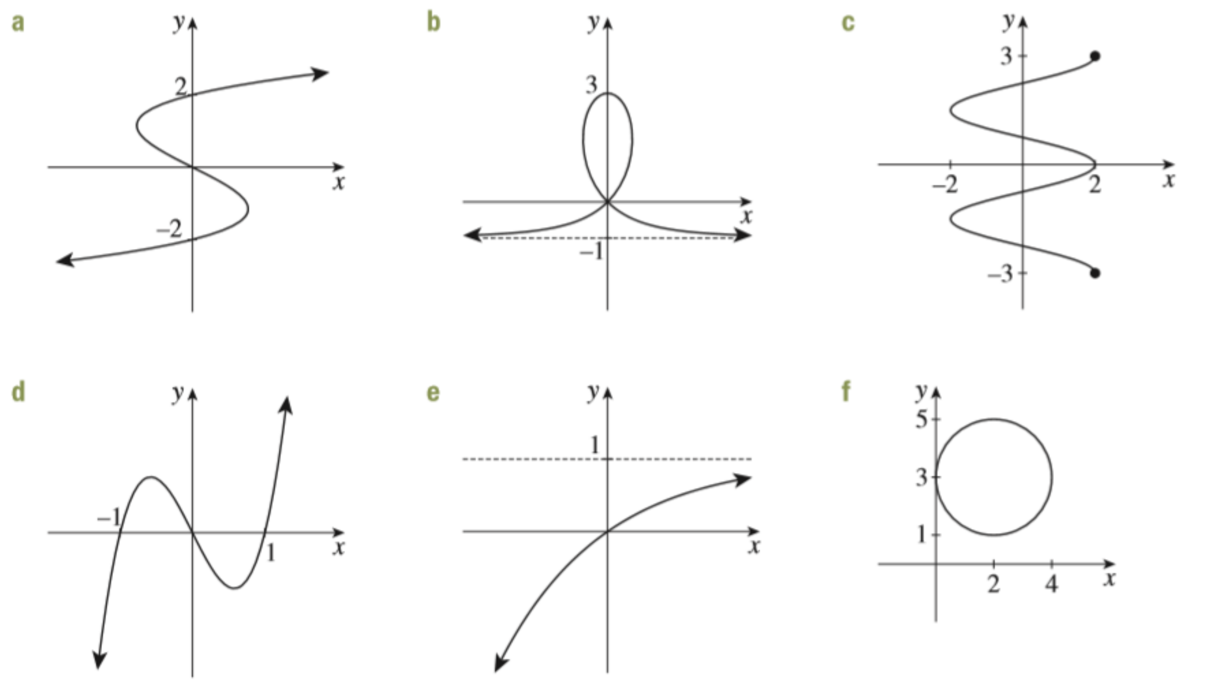
6. Write down the zeroes of each polynomial, use a table of values to test its sign, then sketch it, showing the y-intercept.

a $y = (x + 2)(x + 1)x(x - 1)(x - 2)$

b $y = -(x - 3)^2(x + 2)^2$

c $y = 2x^2(x - 2)^4(x - 4)$

7. Classify each graph as one-to-one, many-to-one, one-to-many or many-to-many



8. Classify each relation as one-to-one, many-to-one, one-to-many or many-to-many.

a $y = 4$ **b** $x = -3$ **c** $x + y = 0$ **d** $x^2 + y^2 = 0$

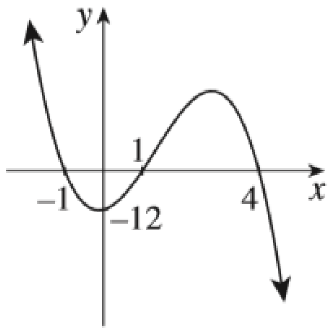
e $x^2 - y^2 = 0$ **f** $x = y^2 - 5y + 6$ **g** $y = x^3 - 7x^2 + 12x$ **h** $y = x^3 + 8$

9. Rewrite each equation with x as the independent variable and then sketch its graph.

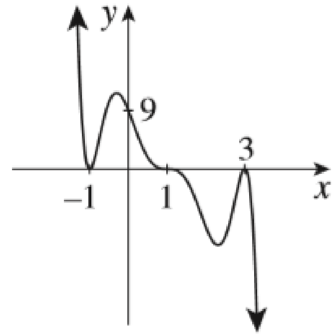
a $xy = \frac{1}{2}$ **b** $xy = -6$

10. These graphs are known to be polynomials, and the second is known to have degree 7. Write down their equations factored into linear factors.

a



b



11. Write down the coordinates of the vertex and the concavity for each parabola. Hence determine the number of x-intercepts and sketch the curves.

a $y = 2(x-3)^2 - 5$ **b** $y = 3 - (x+1)^2$ **c** $y = -3(x+2)^2 - 1$

COSMOS